

Hartwig's triple reverse order law revisited

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Abstract

We extend the classical Hartwig's triple reverse order law for the Moore-Penrose inverse to closed-range bounded linear operators on infinite dimensional Hilbert spaces.

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1 Introduction

If S is a semigroup with the unit 1, and if $a, b \in S$ are invertible, then the equality $(ab)^{-1} = b^{-1}a^{-1}$ is called the reverse order law for the ordinary inverse. It is well-known that the reverse order law does not hold for various classes of generalized inverses.

In this paper we specialize our investigations to the Moore-Penrose inverse of a triple product of closed range bounded linear operators on Hilbert spaces.

Let X, Y, Z be Hilbert spaces, and let $\mathcal{L}(X, Y)$ denote the set of all bounded linear operators from X to Y . We abbreviate $\mathcal{L}(X) = \mathcal{L}(X, X)$. For $A \in \mathcal{L}(X, Y)$ we denote by $\mathcal{N}(A)$ and $\mathcal{R}(A)$, respectively, the null-space and the range of A . An operator $B \in \mathcal{L}(Y, X)$ is an inner inverse of A , if $ABA = A$ holds. In this case A is inner invertible, or relatively regular. It is well-known that A is inner invertible if and only if $\mathcal{R}(A)$ is closed in Y . The Moore-Penrose inverse of $A \in \mathcal{L}(X, Y)$ is the operator $X \in \mathcal{L}(Y, X)$ which satisfies the Penrose equations

$$AXA = A, \quad XAX = X, \quad (AX)^* = AX, \quad (XA)^* = XA.$$

The Moore-Penrose inverse of A exists if and only if $\mathcal{R}(A)$ is closed in Y . If the Moore-Penrose inverse of A exists, then it is unique, and it is denoted by A^\dagger .

The rule $(AB)^\dagger = B^\dagger A^\dagger$ (in the case when A, B, AB have closed ranges) does not hold in general. The equivalence conditions can be found in [7] for complex matrices; see [8], [2] and [3] for closed range bounded linear

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operators on Hilbert spaces; see [9] for Moore-Penrose invertible elements in rings and C^* -algebras.

Notice that the reverse order rule attracts a significant attention (see [1], [4], [6], [10], [11] and [13]).

The classical result of Hartwig [12] deals with the triple reverse order law of the form

$$(ABC)^\dagger = C^\dagger B^\dagger A^\dagger, \quad (1)$$

where A, B, C are matrices. Hartwig establishes several equivalent conditions such that (1) holds, offering a very general proof of the main result. However, one implication in [12] is not valid in infinite dimensional Hilbert spaces, and thus we find it interesting to extend Hartwig's proof in this direction.

We start with some auxiliary results.

Lemma 1.1. *Let $A \in \mathcal{L}(X, Y)$ have a closed range. Then A has the matrix decomposition with respect to the orthogonal decompositions of spaces $X = \mathcal{R}(A^*) \oplus \mathcal{N}(A)$ and $Y = \mathcal{R}(A) \oplus \mathcal{N}(A^*)$:*

$$A = \begin{bmatrix} A_1 & 0 \\ 0 & 0 \end{bmatrix} : \begin{bmatrix} \mathcal{R}(A^*) \\ \mathcal{N}(A) \end{bmatrix} \rightarrow \begin{bmatrix} \mathcal{R}(A) \\ \mathcal{N}(A^*) \end{bmatrix},$$

where A_1 is invertible. Moreover,

$$A^\dagger = \begin{bmatrix} A_1^{-1} & 0 \\ 0 & 0 \end{bmatrix} : \begin{bmatrix} \mathcal{R}(A) \\ \mathcal{N}(A^*) \end{bmatrix} \rightarrow \begin{bmatrix} \mathcal{R}(A^*) \\ \mathcal{N}(A) \end{bmatrix}.$$

The proof of the previous result is straightforward.

Lemma 1.2. [6] *Let $A \in \mathcal{L}(X, Y)$ have a closed range. Let X_1 and X_2 be closed and mutually orthogonal subspaces of X , such that $X = X_1 \oplus X_2$. Let Y_1 and Y_2 be closed and mutually orthogonal subspaces of Y , such that $Y = Y_1 \oplus Y_2$. Then the operator A has the following matrix representations with respect to the orthogonal sums of subspaces $X = X_1 \oplus X_2 = \mathcal{R}(A^*) \oplus \mathcal{N}(A)$, and $Y = \mathcal{R}(A) \oplus \mathcal{N}(A^*) = Y_1 \oplus Y_2$:*

(a)

$$A = \begin{bmatrix} A_1 & A_2 \\ 0 & 0 \end{bmatrix} : \begin{bmatrix} X_1 \\ X_2 \end{bmatrix} \rightarrow \begin{bmatrix} \mathcal{R}(A) \\ \mathcal{N}(A^*) \end{bmatrix},$$

where $D = A_1 A_1^* + A_2 A_2^*$ maps $\mathcal{R}(A)$ into itself and $D > 0$. Also,

$$A^\dagger = \begin{bmatrix} A_1^* D^{-1} & 0 \\ A_2^* D^{-1} & 0 \end{bmatrix}.$$

(b)

$$A = \begin{bmatrix} A_1 & 0 \\ A_2 & 0 \end{bmatrix} : \begin{bmatrix} \mathcal{R}(A^*) \\ \mathcal{N}(A) \end{bmatrix} \rightarrow \begin{bmatrix} Y_1 \\ Y_2 \end{bmatrix},$$

where $D = A_1^*A_1 + A_2^*A_2$ maps $\mathcal{R}(A^*)$ into itself and $D > 0$. Also,

$$A^\dagger = \begin{bmatrix} D^{-1}A_1^* & D^{-1}A_2^* \\ 0 & 0 \end{bmatrix}.$$

Here A_i denotes different operators in any of these two cases.

Lemma 1.3. *Let $A \in \mathcal{L}(X, Y)$ be closed range operator and let P_M be orthogonal projection from Y to closed subspace $\mathcal{R}(M) \subset \mathcal{R}(A)$. Then A^*P_MA has a closed range.*

Proof. According to Lemma 1.1 and Lemma 1.2, operators A and P_M have the following forms:

$$A = \begin{bmatrix} A_1 & 0 \\ A_2 & 0 \end{bmatrix} : \begin{bmatrix} \mathcal{R}(A^*) \\ \mathcal{N}(A) \end{bmatrix} \rightarrow \begin{bmatrix} \mathcal{R}(M) \\ \mathcal{N}(M) \end{bmatrix},$$

$$P_M = \begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix} : \begin{bmatrix} \mathcal{R}(M) \\ \mathcal{N}(M) \end{bmatrix} \rightarrow \begin{bmatrix} \mathcal{R}(M) \\ \mathcal{N}(M) \end{bmatrix}.$$

It is obvious that $A^*P_MA = (P_MA)^*P_MA$, and by using well-known fact that for any bounded linear operator T holds: T^*T has closed range if and only if T has closed range, it is enough to prove that P_MA is closed range operator. From the form of P_MA :

$$P_MA = \begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} A_1 & 0 \\ A_2 & 0 \end{bmatrix} = \begin{bmatrix} A_1 & 0 \\ 0 & 0 \end{bmatrix} : \begin{bmatrix} \mathcal{R}(A^*) \\ \mathcal{N}(A) \end{bmatrix} \rightarrow \begin{bmatrix} \mathcal{R}(M) \\ \mathcal{N}(M) \end{bmatrix},$$

we have $\mathcal{R}(P_MA) = \mathcal{R}(A_1) = A_1(\mathcal{R}(A^*))$, which is closed because A_1 is onto. Indeed, let us suppose A_1 is not onto; this means there is some $y \in \mathcal{R}(M) \setminus \mathcal{R}(A_1)$. Because of $\mathcal{R}(M) \subset \mathcal{R}(A)$, there is some $x \in \mathcal{R}(A^*)$ such that $y = A_1x + A_2x$, provided that $A_2x \neq 0$. Therefore, $\mathcal{R}(M) \ni y - A_1x = A_2x \in \mathcal{N}(M)$, and sum $\mathcal{R}(M) \oplus \mathcal{N}(M)$ is direct, so $A_2x = 0$, which is contradiction. Therefore, A_1 is onto. \square

2 Main result

In this section we extend results due to Hartwig [12] concerning the triple reverse order law for the Moore-Penrose inverse from complex matrices to infinite dimensional settings.

In this section, let X_i , $i = 1, 2, 3, 4$, be arbitrary Hilbert spaces, and let $A \in \mathcal{L}(X_3, X_4)$, $B \in \mathcal{L}(X_2, X_3)$ and $C \in \mathcal{L}(X_1, X_2)$ be bounded linear operators with closed ranges. We use notations in the same way as in [12]:

$$\begin{aligned} M &= ABC, & X &= C^\dagger B^\dagger A^\dagger, \\ E &= A^\dagger A, & F &= CC^\dagger, \\ P &= EBF, & Q &= FB^\dagger E. \end{aligned}$$

Recall that $K \in L(X)$ is EP, if K has a closed range, and $KK^\dagger = K^\dagger K$. The main result is the following theorem.

Theorem 2.1. *Let A, B, C be closed-range operators such that ABC also has a closed range. The following statements are equivalent:*

- (a) $(ABC)^\dagger = C^\dagger B^\dagger A^\dagger$;
- (b) $PQP = P$, $QPQ = Q$, and both of A^*APQ , and $QPCC^*$ are Hermitian;
- (c) $PQP = P$, $QPQ = Q$, and both of A^*APQ , and $QPCC^*$ are EP;
- (d) $PQP = P$, $\mathcal{R}(A^*AP) = \mathcal{R}(Q^*)$, $\mathcal{R}(CC^*P^*) = \mathcal{R}(Q)$;
- (e) $(PQ)^2 = PQ$, $\mathcal{R}(A^*AP) = \mathcal{R}(Q^*)$, $\mathcal{R}(CC^*P^*) = \mathcal{R}(Q)$.

Proof. The proof given by Hartwig stays valid for (a) \Leftrightarrow (b), (b) \Rightarrow (c), (c) \Rightarrow (d) and (d) \Rightarrow (e). The only case which does not hold in general, is actually the implication (e) \Rightarrow (b), which involves properties of the matrix rank. Thus, this part of the proof is not applicable to operators on infinite dimensional Hilbert space.

To complete the proof, we will prove (e) \Rightarrow (a) in a different way, using properties of operator matrices.

Using Lemma 1.1 we conclude that the operator C has the following matrix form:

$$C = \begin{bmatrix} C_1 & 0 \\ 0 & 0 \end{bmatrix} : \begin{bmatrix} \mathcal{R}(C^*) \\ \mathcal{N}(C) \end{bmatrix} \rightarrow \begin{bmatrix} \mathcal{R}(C) \\ \mathcal{N}(C^*) \end{bmatrix},$$

where C_1 is invertible. Then

$$C^\dagger = \begin{bmatrix} C_1^{-1} & 0 \\ 0 & 0 \end{bmatrix} : \begin{bmatrix} \mathcal{R}(C) \\ \mathcal{N}(C^*) \end{bmatrix} \rightarrow \begin{bmatrix} \mathcal{R}(C^*) \\ \mathcal{N}(C) \end{bmatrix}.$$

From Lemma 1.2 it follows that the operator B has the following matrix form:

$$B = \begin{bmatrix} B_1 & B_2 \\ 0 & 0 \end{bmatrix} : \begin{bmatrix} \mathcal{R}(C) \\ \mathcal{N}(C^*) \end{bmatrix} \rightarrow \begin{bmatrix} \mathcal{R}(B) \\ \mathcal{N}(B^*) \end{bmatrix},$$

where $G = B_1B_1^* + B_2B_2^*$ is invertible and positive in $\mathcal{L}(\mathcal{R}(B))$. Then

$$B^\dagger = \begin{bmatrix} B_1^*G^{-1} & 0 \\ B_2^*G^{-1} & 0 \end{bmatrix}.$$

From Lemma 1.2 it also follows that the operator A has the following matrix form:

$$A = \begin{bmatrix} A_1 & A_2 \\ 0 & 0 \end{bmatrix} : \begin{bmatrix} \mathcal{R}(B) \\ \mathcal{N}(B^*) \end{bmatrix} \rightarrow \begin{bmatrix} \mathcal{R}(A) \\ \mathcal{N}(A^*) \end{bmatrix},$$

where $D = A_1A_1^* + A_2A_2^*$ is invertible and positive in $\mathcal{L}(\mathcal{R}(A))$. Then

$$A^\dagger = \begin{bmatrix} A_1^*D^{-1} & 0 \\ A_2^*D^{-1} & 0 \end{bmatrix}.$$

Let us find the expressions for the operators M , X , E , F , P and Q . It is easy to find that:

$$M = ABC = \begin{bmatrix} A_1B_1C_1 & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} M_1 & 0 \\ 0 & 0 \end{bmatrix}, \quad M_1 = A_1B_1C_1;$$

$$X = C^\dagger B^\dagger A^\dagger = \begin{bmatrix} C_1^{-1}B_1^*G^{-1}A_1^*D^{-1} & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} X_1 & 0 \\ 0 & 0 \end{bmatrix},$$

$$X_1 = C_1^{-1}B_1^*G^{-1}A_1^*D^{-1};$$

$$E = A^\dagger A = \begin{bmatrix} A_1^*D^{-1}A_1 & A_1^*D^{-1}A_2 \\ A_2^*D^{-1}A_1 & A_2^*D^{-1}A_2 \end{bmatrix}; \quad F = CC^\dagger = \begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix};$$

$$P = EBF = \begin{bmatrix} A_1^*D^{-1}A_1B_1 & 0 \\ A_2^*D^{-1}A_1B_1 & 0 \end{bmatrix} = \begin{bmatrix} A_1^*D^{-1}M_1C_1^{-1} & 0 \\ A_2^*D^{-1}M_1C_1^{-1} & 0 \end{bmatrix};$$

$$Q = FB^\dagger E = \begin{bmatrix} B_1^* G^{-1} A_1^* D^{-1} A_1 & B_1^* G^{-1} A_1^* D^{-1} A_2 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} C_1 X_1 A_1 & C_1 X_1 A_2 \\ 0 & 0 \end{bmatrix}.$$

It will be convenient to compute here matrix forms for some expressions appearing in the rest of the proof:

$$PQ = \begin{bmatrix} A_1^* D^{-1} M_1 X_1 A_1 & A_1^* D^{-1} M_1 X_1 A_2 \\ A_2^* D^{-1} M_1 X_1 A_1 & A_2^* D^{-1} M_1 X_1 A_2 \end{bmatrix};$$

$$QP = \begin{bmatrix} C_1 X_1 M_1 C_1^{-1} & 0 \\ 0 & 0 \end{bmatrix};$$

$$A^* AP = \begin{bmatrix} A_1^* M_1 C_1^{-1} & 0 \\ A_2^* M_1 C_1^{-1} & 0 \end{bmatrix};$$

$$CC^* P^* = \begin{bmatrix} C_1 M_1^* D^{-1} A_1 & C_1 M_1^* D^{-1} A_2 \\ 0 & 0 \end{bmatrix};$$

$$(PQ)^2 = \begin{bmatrix} A_1^* D^{-1} M_1 X_1 M_1 X_1 A_1 & A_1^* D^{-1} M_1 X_1 M_1 X_1 A_2 \\ A_2^* D^{-1} M_1 X_1 M_1 X_1 A_1 & A_2^* D^{-1} M_1 X_1 M_1 X_1 A_2 \end{bmatrix}.$$

Now, we will find equivalent expressions for the conditions (a) and (e) in the terms of the components of the operators A, B and C .

(a) : This is equivalent to $(A_1 B_1 C_1)^\dagger = C_1^{-1} B_1^* G^{-1} A_1^* D^{-1}$, or $M_1^\dagger = X_1$.

(e) : This is equivalent to the following three expressions:

$$(e.1) \Leftrightarrow A_i^* D^{-1} (M_1 X_1)^2 A_j = A_i^* D^{-1} M_1 X_1 A_j, \quad \text{for all } i, j \in \{1, 2\};$$

$$(e.2) \Leftrightarrow \mathcal{R} \left(\begin{bmatrix} A_1^* M_1 C_1^{-1} & 0 \\ A_2^* M_1 C_1^{-1} & 0 \end{bmatrix} \right) = \mathcal{R} \left(\begin{bmatrix} A_1^* X_1^* C_1^* & 0 \\ A_2^* X_1^* C_1^* & 0 \end{bmatrix} \right);$$

$$(e.3) \Leftrightarrow \mathcal{R} \left(\begin{bmatrix} C_1 M_1^* D^{-1} A_1 & C_1 M_1^* D^{-1} A_2 \\ 0 & 0 \end{bmatrix} \right) = \mathcal{R} \left(\begin{bmatrix} C_1 X_1 A_1 & C_1 X_1 A_2 \\ 0 & 0 \end{bmatrix} \right).$$

Recall that we prove the implication (e) \implies (a).

Now, if we premultiply (e.1) by A_i , and use summation over $i = 1, 2$ we yield $(M_1 X_1)^2 A_j = M_1 X_1 A_j$, for $j = 1, 2$. If we now postmultiply last expression by A_j^* and add them, we have $(M_1 X_1)^2 = M_1 X_1$. Therefore:

$$(e.1) \implies (M_1 X_1)^2 = M_1 X_1. \quad (2)$$

On the other hand, (e.2) is equivalent to:

$$\mathcal{R}(A_i^* M_1 C_1^{-1}) = \mathcal{R}(A_i^* X_1^* C_1^*), \quad i = 1, 2.$$

Again, if A_i acts on both sides, and we add them, we obtain:

$$\mathcal{R}(M_1 C_1^{-1}) = \mathcal{R}(X_1^* C_1^*).$$

Hence, we have

$$\mathcal{R}(M_1) = \mathcal{R}(X_1^*),$$

which implies $M_1 M_1^\dagger = X_1^\dagger X_1$. Therefore,

$$(e.2) \Rightarrow M_1 M_1^\dagger = X_1^\dagger X_1.$$

Let us now write (e.3) as:

$$\mathcal{N} \left(\begin{bmatrix} A_1^* D^{-1} M_1 C_1^* & 0 \\ A_2^* D^{-1} M_1 C_1^* & 0 \end{bmatrix} \right) = \mathcal{N} \left(\begin{bmatrix} A_1^* X_1^* C_1^* & 0 \\ A_2^* X_1^* C_1^* & 0 \end{bmatrix} \right).$$

Notice that

$$\mathcal{N} \left(\begin{bmatrix} A_1^* D^{-1} M_1 C_1^* & 0 \\ A_2^* D^{-1} M_1 C_1^* & 0 \end{bmatrix} \right) = \left\{ \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} : \begin{bmatrix} A_1^* D^{-1} M_1 C_1^* & 0 \\ A_2^* D^{-1} M_1 C_1^* & 0 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \right\},$$

and we conclude:

$$\mathcal{N} \left(\begin{bmatrix} A_1^* D^{-1} M_1 C_1^* & 0 \\ A_2^* D^{-1} M_1 C_1^* & 0 \end{bmatrix} \right) = \left(\mathcal{N}(A_1^* D^{-1} M_1 C_1^*) \cap \mathcal{N}(A_2^* D^{-1} M_1 C_1^*) \right) \oplus \mathcal{N}(C^*),$$

which is further equal (easy to see) to

$$\mathcal{N}(M_1 C_1^*) \oplus \mathcal{N}(C^*).$$

With a little effort we find

$$\begin{aligned} \mathcal{N} \left(\begin{bmatrix} A_1^* X_1^* C_1^* & 0 \\ A_2^* X_1^* C_1^* & 0 \end{bmatrix} \right) &= \left(\mathcal{N}(A_1^* X_1^* C_1^*) \cap \mathcal{N}(A_2^* X_1^* C_1^*) \right) \oplus \mathcal{N}(C^*) = \\ &= \mathcal{N}(X_1^* C_1^*) \oplus \mathcal{N}(C^*). \end{aligned}$$

Hence, the condition (e.3) implies:

$$\mathcal{N}(M_1 C_1^*) = \mathcal{N}(X_1^* C_1^*),$$

which is the same as $\mathcal{R}(C_1M_1^*) = \mathcal{R}(C_1X_1)$, or $\mathcal{R}(M_1^*) = \mathcal{R}(X_1)$, or even further: $M_1^\dagger M_1 = X_1 X_1^\dagger$.

Since we intend to prove (e) \Rightarrow (a), it is enough to prove the following implication:

$$\left((M_1 X_1)^2 = M_1 X_1, \quad M_1 M_1^\dagger = X_1^\dagger X_1, \quad M_1^\dagger M_1 = X_1 X_1^\dagger \right) \Rightarrow M_1^\dagger = X_1.$$

The following completes the proof:

$$\begin{aligned} M_1 &= M_1 X_1 X_1^\dagger = M_1 X_1 X_1^\dagger X_1 X_1^\dagger = M_1 X_1 M_1 M_1^\dagger X_1^\dagger = \\ &= M_1 X_1 M_1 X_1 X_1^\dagger M_1^\dagger X_1^\dagger = M_1 X_1 X_1^\dagger M_1^\dagger X_1^\dagger = \\ &= M_1 M_1^\dagger X_1^\dagger = X_1^\dagger X_1 X_1^\dagger = X_1^\dagger. \end{aligned}$$

For the sake of completeness, we remark that operators A^*APQ and $QPCC^*$ from part (c) of our Theorem have closed ranges. It immediately follows from Lemma 1.3 because:

$$A^*APQ = A^*MM^\dagger A = A^*P_{\mathcal{R}(M)}A, \quad QPCC^* = CM^\dagger MC^* = CP_{\mathcal{R}(M)}C^*.$$

□

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